Extended canonical ideals and Goto rings

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Introduction

Question 1.1

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

Problem 1.2

Find new and interesting classes of rings which fill in a gap between Gorenstein and Cohen-Macaulay rings, so as to stratify Cohen-Macaulay rings.

Problem 1.3

Find new classes of CM rings which may not be Gorenstein, but sufficiently good next to Gorenstein rings.

Almost Gorenstein rings

- [Barucci-Fröberg, 1997]

  $\cdots$ one-dimensional analytically unramified local rings
- [Goto-Matsuoka-Phuong, 2013]

  $\cdots$ one-dimensional CM local rings
- [Goto-Takahashi-Taniguchi, 2015]

  $\cdots$ higher-dimensional CM local/graded rings

Generalization of AGL rings

- 2-almost Gorenstein local rings (Chau-Goto-Kumashiro-Matsuoka)
- Generalized Gorenstein local rings (Goto-Kumashiro)

Question 1.4

Can we construct a new theory that can be understood AGL, 2-AGL and GGL rings in a unified manner?

Extended canonical ideals

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$, $\exists\, \mathrm{K}_A$, and $|A/\mathfrak{m}| = \infty$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$

Recall that $\exists$ a canonical ideal $\iff$ $A_{\mathfrak{p}}$ is Gorenstein for $\forall\, \mathfrak{p} \in \operatorname{Min} A$.

For ideals J and Q with $Q \subseteq J$,

- Q is a reduction of J, if $J^{r+1} = QJ^r$ for $\exists\, r \geq 0$
- $\mathrm{red}_Q(J) = \min\{r \geq 0 \mid J^{r+1} = QJ^r\}$.

Definition 2.1

(1) A parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of A satisfies the condition $(\sharp)$, if $a_1 \in I$ and Q is a reduction of $Q + I$.

(2) An ideal J is called an extended canonical ideal of A, if $J = I + Q$ for some parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ satisfying $(\sharp)$.

Example 2.2

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$, let

- $A = k[[X_1, X_2, \ldots, X_\ell, V_1, V_2, \ldots, V_{\ell-1}]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2+V_1 & \cdots & X_{\ell-1}+V_{\ell-2} & X_\ell+V_{\ell-1} \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$
- $I = (X_1^n, X_2, \ldots, X_{\ell-1})A$
- $Q = (X_1^n, V_1, \ldots, V_{\ell-1})A$.

Then A is a CM local ring admitting the extended canonical ideal $J = I + Q$.

- When $d = 1$, extended canonical ideals $=$ canonical ideals.
- When $d \geq 2$, JA' is a canonical ideal of $A' = A/(a_2, a_3, \ldots, a_d)$.
- An extended canonical ideal exists.

There exist integers $\{\mathrm{e}_i(J)\}_{0 \leq i \leq d}$ s.t.

$$\ell_A(A/J^{n+1}) = \mathrm{e}_0(J)\binom{n+d}{d} - \mathrm{e}_1(J)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(J) \text{ for } \forall n \gg 0.$$

- $\mathrm{e}_1(J) \geq \mathrm{e}_0(J) - \ell_A(A/J)$
- $\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_A(A/J) \iff J^2 = QJ$.

  When this is the case,

  (1) $\mathrm{gr}_J(A) = \bigoplus_{i \geq 0} J^i/J^{i+1}$ and $\mathcal{F}(J) = \bigoplus_{i \geq 0} J^i/\mathfrak{m}J^i$ are CM

  (2) $\mathcal{R}(J) = \bigoplus_{i \geq 0} J^i$ is CM, if $d \geq 2$.

- Sally characterized J with $\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_A(A/J) + 1$ and $\mathrm{e}_2(J) \neq 0$.
- Vasconcelos introduced $\mathcal{S}_Q(J) = \bigoplus_{i \geq 1} J^{i+1}/JQ^i$, recovered Sally's results, and made further progress, e.g.,

$$\mathrm{rank}\, \mathcal{S}_Q(J) = \mathrm{e}_1(J) - \mathrm{e}_0(J) + \ell_A(A/J).$$

- Goto, Nishida, and Ozeki established the theory of $\mathrm{rank}\, \mathcal{S}_Q(J) = 1$.

Whereas they considered any $\mathfrak{m}$-primary ideals, we focus on extended canonical ideals and raise the rank of the Sally modules.

Goto rings

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$, $\exists\, \mathrm{K}_A$, and $|A/\mathfrak{m}| = \infty$
- $I \subsetneq A$ an ideal of A s.t. $I \cong \mathrm{K}_A$, and $n \geq 0$ an integer

Definition 3.1

Let $Q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal of A. We say that A is an n-Goto ring with respect to Q, if

$$a_1 \in I, \quad J^3 = QJ^2, \quad \text{and} \quad \ell_A(J^2/QJ) = n$$

where $J = I + Q$. The ring A is called n-Goto, if $\exists$ a parameter ideal Q of A s.t. A is n-Goto with respect to Q.

- A is 0-Goto $\iff$ A is Gorenstein
- A is 1-Goto $\iff$ A is non-Gorenstein AGL
- A is 2-Goto $\iff$ A is 2-AGL, provided $d = 1$
- A is $\ell_A(A/\mathfrak{a})$-Goto $\Longleftarrow$ A is GGL with respect to $\mathfrak{a}$, where $\sqrt{\mathfrak{a}} = \mathfrak{m}$.

Example 3.2

Let k be a field.

(1) $k[[t^3, t^{3n+1}, t^{3n+2}]]$ is n-Goto and is an integral domain.

(2) $k[[t^3, t^{3n+1}, t^{3n+2}]] \times_k k[[t]]$ is n-Goto, reduced, but not an integral domain.

(3) $k[[t^3, t^{3n+1}, t^{3n+2}]] \ltimes k[[t]]$ is n-Goto and is not reduced.

Recall that $x \in J$ is super-regular, if $xt \in \mathcal{R}(J)$ is a NZD on $\mathrm{gr}_J(A)$.

Theorem 3.3

Suppose that $d \geq 2$. Let $J = I + Q$ be an extended canonical ideal of A and set $\mathfrak{q} = (a_2, a_3, \dots, a_d)$. Let $x \in \mathfrak{q} \setminus \mathfrak{m}\mathfrak{q}$ be super-regular. Then TFAE.

(1) *A is an n-Goro ring with respect to Q.*

(2) *$A/(x)$ is an n-Goto ring with respect to $Q/(x)$.*

Example 3.4

Let k be a field. For any $\ell \geq 3$, $m \geq n \geq 2$,

$$A = k[[X_1, X_2, \ldots, X_\ell, V_1, V_2, \ldots, V_{\ell-1}]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2+V_1 & \cdots & X_{\ell-1}+V_{\ell-2} & X_\ell+V_{\ell-1} \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim A = \ell$ and $\mathrm{r}(A) = \ell - 1$.

- $(A_1, \mathfrak{m}_1)$ a CM local ring with $\dim A_1 = d$
- $\varphi : A \to A_1$ a flat local map s.t. $A_1/\mathfrak{m}A_1$ is Gorenstein
- Q a parameter ideal of A with $(\sharp)$

Then IA_1 is a canonical ideal of A_1 and QA_1 is a parameter ideal of A_1 with $(\sharp)$.

Theorem 3.5

TFAE.

(1) A_1 *is* n*-Goto with respect to* QA_1.

(2) $\exists\, m > 0$ *s.t.* $m \mid n$, A *is* m*-Goto with respect to* Q, *and* $\ell_{A_1}(A_1/\mathfrak{m}A_1) = \frac{n}{m}$.

Example 3.6 (cf. Chau-Goto-Kumashiro-Matsuoka)

Let $A_1 = A[X]/(X^n + \alpha_1 X^{n-1} + \cdots + \alpha_n)$ $(n \geq 1,\ \alpha_i \in \mathfrak{m})$. Then

- A_1 is a flat local A-algebra with $\mathfrak{m}_1 = \mathfrak{m}A_1 + XA_1$
- $A_1/\mathfrak{m}A_1 = (A/\mathfrak{m})[X]/(X^n)$ is an Artinian Gorenstein ring
- $\ell_{A_1}(A_1/\mathfrak{m}A_1) = n$.

Hence, if $n \geq 2$ is a prime integer, then

$$A_1 \text{ is } n\text{-Goto with respect to } QA_1 \iff A \text{ is non-Gorenstein AGL}$$

where Q is a parameter ideal of A with $(\sharp)$.

One-dimensional Goto rings

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$, $\exists \mathrm{K}_R$, and $|R/\mathfrak{m}| = \infty$
- I a canonical ideal of R, $n \geq 0$ an integer
- $Q = (a)$ a parameter ideal of R s.t. Q is a reduction of I
- $K = \dfrac{I}{a} = \left\{ \dfrac{x}{a} \;\middle|\; x \in I \right\} \subseteq \mathrm{Q}(R)$

- $K \cong \mathrm{K}_R$ and $R \subseteq K \subseteq \overline{R} \subseteq \mathrm{Q}(R)$
- R is n-Goto $\iff$ $K^3 = K^2$ and $\ell_R(K^2/K) = n$

Example 4.1

The ring $R = k[[H]] = k[[t^h \mid h \in H]]$ $(\subseteq k[[t]])$ is an n-Goto ring, where

- $H = \langle 3, 3n+1, 3n+2 \rangle$ $(n \geq 1)$
- $H = \langle e, \{en - e + i\}_{3 \leq i \leq e-1}, en+1, en+2 \rangle$ $(n \geq 2,\ e \geq 4)$.

We set $R^I = \bigcup_{n\geq 0} [I^n : I^n]$ for a regular ideal I of R.

Theorem 4.2

Suppose that R has minimal multiplicity, $R^\mathfrak{m}$ is a local ring, and $R/\mathfrak{m} \cong R^\mathfrak{m}/\mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal ideal of $R^\mathfrak{m}$. Then TFAE for $n \geq 1$.

(1) *R is an n-Goto ring*

(2) *$R^\mathfrak{m}$ is an $(n-1)$-Goto ring.*

Example 4.3

Let k be a field. Then

- $R = k[[t^5, t^{13}, t^{14}, t^{16}, t^{17}]]$ is 3-Goto.
- $R_1 = R^\mathfrak{m} = \bigcup_{n\geq 0} [\mathfrak{m}^n : \mathfrak{m}^n] = \mathfrak{m} : \mathfrak{m} = k[[t^5, t^8, t^9, t^{11}, t^{12}]]$ is 2-Goto.
- $R_2 = (R_1)^{\mathfrak{m}_1} = \bigcup_{n\geq 0} [\mathfrak{m}_1^n : \mathfrak{m}_1^n] = \mathfrak{m}_1 : \mathfrak{m}_1 = k[[t^3, t^4, t^5]]$ is 1-Goto.
- $R_3 = (R_2)^{\mathfrak{m}_2} = \bigcup_{n\geq 0} [\mathfrak{m}_2^n : \mathfrak{m}_2^n] = \mathfrak{m}_2 : \mathfrak{m}_2 = k[[t]] = \overline{R}$ is 0-Goto.

- $(S, \mathfrak{n})$ a CM local ring with $\dim S = 1$ and $k = R/\mathfrak{m} = S/\mathfrak{n}$
- $f : R \to k$, $g : S \to k$ canonical maps
- $R \times_k S = \{(a, b) \in R \times S \mid f(a) = g(b)\} \subseteq R \times S$

Then $R \times_k S$ is a CM local ring with $\dim(R \times_k S) = 1$. Moreover

$$R \times_k S \text{ is Gorenstein} \iff R \text{ and } S \text{ are DVRs.}$$

Theorem 4.4

Suppose that $\exists\, \mathrm{K}_{(R \times_k S)}$ and $\mathrm{Q}(R \times_k S)$ is Gorenstein. Then TFAE for $n \geq 2$.

(1) *$R \times_k S$ is an n-Goto ring.*

(2) *One of the following conditions holds.*

  (i) *R is Gorenstein and S is n-Goto.*
  (ii) *R is n-Goto and S is Gorenstein.*
  (iii) *R is p-Goto and S is q-Goto for $\exists\, p, q > 0$ s.t. $n + 1 = p + q$.*

Hence, if R is n-Goto and S is 2-Goto, then $R \times_k S$ is $(n + 1)$-Goto.

Recall that R is n-Goto $\iff$ $K^2 = K^3$ and $\ell_R(K^2/K) = n$.

Lemma 4.5

Suppose $\mathrm{r}(R) = 2$. For each $n \geq 1$, we have

$$R \text{ is } n\text{-Goto} \iff K^2 = K^3 \text{ and } \ell_R(K/R) = n.$$

- $R = k[[t^{a_1}, t^{a_2}, t^{a_3}]]$, where $0 < a_1, a_2, a_3 \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, a_3) = 1$
- R is not a Gorenstein ring
- $\varphi : k[[X, Y, Z]] \to R$ the k-algebra map s.t.

$$\varphi(X) = t^{a_1}, \ \varphi(Y) = t^{a_2}, \text{ and } \varphi(Z) = t^{a_3}$$

Then

$$\mathrm{Ker}\,\varphi = \mathrm{I}_2 \begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix} \text{ for } \exists\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0.$$

By setting $b = a_1\alpha - a_2\beta'$ $(= a_2\beta - a_3\gamma' = a_3\gamma - a_1\alpha')$, we get

$$\ell_R(K/R) = \begin{cases} \alpha\beta\gamma & (b < 0) \\ \alpha'\beta'\gamma' & (b > 0) \end{cases}.$$

Theorem 4.6

Suppose that R is not Gorenstein. Then TFAE for $n \geq 1$, where $H = \langle a_1, a_2, a_3 \rangle$.

(1) $R = k[[H]]$ *is an n-Goto ring.*

(2) $3 \cdot |b| \in H$, *and* $n = \alpha\beta\gamma$ (*resp.* $n = \alpha'\beta'\gamma'$) *if* $b < 0$ (*resp.* $b > 0$).

Example 4.7

Let $R = k[[t^7, t^{10}, t^{22}]]$. The k-algebra map $\varphi : k[[X, Y, Z]] \to R$ defined by

$$\varphi(X) = t^{10}, \ \ \varphi(Y) = t^7, \ \ \text{and} \ \ \varphi(Z) = t^{22}$$

induces

$$R \cong k[[X, Y, Z]]/\mathrm{I}_2 \begin{pmatrix} X^2 & Y^2 & Z \\ Y^4 & Z & X^3 \end{pmatrix}.$$

Then $b = a_1\alpha - a_2\beta' = 10 \cdot 2 - 7 \cdot 4 = -8 < 0$. Hence, $3 \cdot |b| = 24 \in H$ and

$$\ell_R(K/R) = 2 \cdot 2 \cdot 1 = 4,$$

so that R is a 4-Goto ring.

Corollary 4.8

Suppose that $\mathrm{e}(R) = 3$ *and* R *has minimal multiplicity. Then TFAE for* $n \geq 1$*, where* $H = \langle a_1, a_2, a_3 \rangle$.

(1) $R = k[[H]]$ *is an* n*-Goto ring.*

(2) $H = \langle 3, 2n + \alpha, n + 2\alpha \rangle$ *for* $\exists\, \alpha \geq n + 1$ *s.t.* $\alpha \not\equiv n \mod 3$.

When this is the case,

$$R \cong k[[X, Y, Z]]/\mathrm{I}_2\begin{pmatrix} X^n & Y & Z \\ Y & Z & X^\alpha \end{pmatrix} \quad \text{or} \quad R \cong k[[X, Y, Z]]/\mathrm{I}_2\begin{pmatrix} X^\alpha & Y & Z \\ Y & Z & X^n \end{pmatrix}.$$

Minimal free resolutions

- $(T, \mathfrak{n})$ a RLR with $\dim T = \ell \geq 3$, $\mathfrak{a} \subsetneq T$ an ideal of T s.t. $\mathfrak{a} \subseteq \mathfrak{n}^2$, $n \geq 2$
- $R = T/\mathfrak{a}$ is a CM local ring with $\dim R = 1$, $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$
- K a fractional canonical ideal of R, $\mathfrak{c} = R : R[K]$

Suppose R is an n-Goto ring and $v(R/\mathfrak{c}) = 1$. As $\ell_R(R/\mathfrak{c}) = \ell_R(K^2/K) = n$,

$$\exists\, x_1, x_2, \ldots, x_\ell \in \mathfrak{m} \ \text{ s.t. } \ \mathfrak{m} = (x_1, x_2, \ldots, x_\ell) \quad \text{and} \quad \mathfrak{c} = (x_1^n, x_2, \ldots, x_\ell).$$

By setting $I_i = (x_1^i, x_2, \ldots, x_\ell)$ $(1 \leq i \leq n)$, we have

$$R : K = \mathfrak{c} = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m} \quad \text{and}$$

$$K/R \cong \bigoplus_{i=1}^{n} (R/I_i)^{\oplus \ell_i} \text{ for } \exists\, \ell_n > 0,\ \exists\, \ell_i \geq 0\ (1 \leq i \leq n-1).$$

Write $K = R + \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} R \cdot f_{ij}$ s.t. $(R/I_i)^{\oplus \ell_i} \cong \sum_{j=1}^{\ell_i} (R/\mathfrak{c}) \cdot \overline{f_{ij}}$ in K/R.

Choose $X_i \in \mathfrak{n}$ s.t. $x_i = \overline{X_i}$ in R.

Theorem 5.1

If $R = T/\mathfrak{a}$ is n-Goto and $v(R/\mathfrak{c}) = 1$, then $F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \to 0$ gives a minimal free presentation of K, where $\mathbb{N} = \left[\, -1 \;\; f_{n1} \cdots f_{n\ell_n} \;\; f_{n-1,1} \cdots f_{n-1,\ell_{n-1}} \;\; \cdots \;\; f_{11} \cdots f_{1\ell_1} \,\right]$ and

$$\mathbb{M} = \left[\begin{array}{cccccccccc}
a^{(n)}_{11}\, a^{(n)}_{12} \cdots a^{(n)}_{1\ell} & \cdots & a^{(n)}_{\ell_n 1}\, a^{(n)}_{\ell_n 2} \cdots a^{(n)}_{\ell_n \ell} & & \cdots & & a^{(1)}_{11}\, a^{(1)}_{12} \cdots a^{(1)}_{1\ell} & \cdots & a^{(1)}_{\ell_n 1}\, a^{(1)}_{\ell_n 2} \cdots a^{(1)}_{\ell_n \ell} & c_1\, c_2 \cdots c_q \\
X_1^n X_2 \cdots X_\ell & & & & & & & & & 0 \\
 & \ddots & & & & & & & & 0 \\
 & & X_1^n X_2 \cdots X_\ell & & & & & & & \vdots \\
 & & & X_1^{n-1} X_2 \cdots X_\ell & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & X_1^{n-1} X_2 \cdots X_\ell & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & X_1 X_2 \cdots X_\ell & & & \\
 & & & & & & & \ddots & & 0 \\
 & & & & & & & & X_1 X_2 \cdots X_\ell & 0
\end{array}\right]$$

Moreover, one has

$$\mathfrak{a} = \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \mathrm{I}_2 \begin{pmatrix} a^{(i)}_{j1} & a^{(i)}_{j2} & \cdots & a^{(i)}_{j\ell} \\ X_1^i & X_2 & \cdots & X_\ell \end{pmatrix} + (c_1, c_2, \ldots, c_q).$$

Example 5.2

Let $\varphi : T = k[[X, Y, Z, W]] \to R = k[[t^4, t^{11}, t^{13}, t^{14}]]$ be the k-algebra map s.t.

$$\varphi(X) = t^4,\ \varphi(Y) = t^{11},\ \varphi(Z) = t^{13},\ \text{and } \varphi(W) = t^{14}.$$

Then $K = R + Rt + Rt^3$ is a fractional canonical ideal of R. Hence, $K^2 = K^3$ and $\ell_R(K^2/K) = 3$, so that R is a 3-Goto ring.

The minimal free presentation of K is given by $F_1 \xrightarrow{\mathbb{M}} F_0 \longrightarrow K \longrightarrow 0$, where

$$\mathbb{M} = \begin{bmatrix} Z & -X^3 & -W & -XY & Y & W & X^4 & XZ \\ X^3 & Y & Z & W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X^2 & Y & Z & W \end{bmatrix}.$$

Hence

$$\operatorname{Ker}\varphi = \mathrm{I}_2\begin{pmatrix} Z & -X^3 & -W & -XY \\ X^3 & Y & Z & W \end{pmatrix} + \mathrm{I}_2\begin{pmatrix} Y & W & X^4 & XZ \\ X^2 & Y & Z & W \end{pmatrix}.$$

Theorem 5.3

Let $X_1, X_2, \ldots, X_\ell \in \mathfrak{n}$ *be a regular sop of* T *and assume* K *has a presentation of the form*

$$F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \longrightarrow 0$$

where $\mathbb{M}$ *and* $\mathbb{N}$ *are the matrices of the form stated in Theorem 5.1, satisfying*

- $a_{ij}^{(n)} \in J_n \ (1 \le i \le \ell_n,\ 1 \le j \le \ell)$
- $a_{ij}^{(k)} \in J_n \ (1 \le k \le n-1,\ 1 \le i \le \ell_k,\ 2 \le j \le \ell)$
- $a_{i1}^{(k)} \in J_k \ (1 \le k \le n-1,\ 1 \le i \le \ell_k)$

where $J_i = (X_1^i, X_2, \ldots, X_\ell) \ (1 \le i \le n)$. *Then* R *is n-Goto and* $v(R/\mathfrak{c}) = 1$.

Example 5.4

Let k be a field. For any $\ell \ge 3$, $m \ge n \ge 2$,

$$R = k[[X_1, X_2, \ldots, X_\ell]]/\mathrm{I}_2 \begin{pmatrix} X_1^n & X_2 & \cdots & X_{\ell-1} & X_\ell \\ X_2 & X_3 & \cdots & X_\ell & X_1^m \end{pmatrix}$$

is an n-Goto ring with $\dim R = 1$ and $\mathrm{r}(R) = \ell - 1$.

Sally modules and Goto rings

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A > 0$, $\exists \mathrm{K}_A$, $|A/\mathfrak{m}| = \infty$, $n \geq 0$
- Q a parameter ideal of A with $(\sharp)$
- $J = I + Q$ an extended canonical ideal of A
- $\mathcal{B} = \mathcal{F}(Q) \cong A/\mathfrak{m} \otimes_{A/Q} \mathrm{gr}_Q(A) \cong (A/\mathfrak{m})[X_1, X_2, \ldots, X_d]$

Fact (GMP, CGKM, Goto-Isobe-Kumashiro-Taniguchi)

- A is Gorenstein $\iff$ $\exists\, Q$ with $(\sharp)$ s.t. $\mathcal{S}_Q(J) = (0)$
- A is non-Gorenstein AGL $\iff$ $\exists\, Q$ with $(\sharp)$ s.t. $\mathcal{S}_Q(J) \cong \mathcal{B}(-1)$
- When $d = 1$, A is 2-AGL $\iff \exists\, 0 \to \mathcal{B}(-1) \to \mathcal{S}_Q(J) \to \mathcal{B}(-1) \to 0$
- A is non-Gorenstein GGL $\implies$ $\exists \sqrt{\mathfrak{a}} = m$ and $\exists\, Q$ with $(\sharp)$ s.t.
  $\mathcal{S}_Q(J) \cong [\mathcal{R}(Q)/\mathfrak{a}\mathcal{R}(Q)]\,(-1)$.

Theorem 6.1

Let Q be a parameter ideal with $(\sharp)$. Then TFAE for $n \geq 1$.

(1) *A is n-Goto with respect to Q.*

(2) $\mathcal{S}_Q(J) = \mathcal{R}(Q)\,[\mathcal{S}_Q(J)]_1$ *and* $\operatorname{rank} \mathcal{S}_Q(J) = n$.

(3) $0 \leq \exists\, \ell < n$, $\exists\, s_i \geq 1$ *s.t.* $n = \sum_{i=0}^{\ell} s_i$ *and* $\mathfrak{m}^{\ell}\mathcal{S}_Q(J) \cong \mathcal{B}(-1)^{\oplus s_0}$, *and if* $\ell > 0$, $\exists$ *exact sequences*

$$
\begin{array}{c}
0 \to \mathcal{B}(-1)^{\oplus s_0} \to \mathfrak{m}^{\ell-1}\mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_1} \to 0 \\
0 \to \mathfrak{m}^{\ell-1}\mathcal{S}_Q(J) \to \mathfrak{m}^{\ell-2}\mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_2} \to 0 \\
\vdots \\
0 \to \mathfrak{m}\mathcal{S}_Q(J) \to \mathcal{S}_Q(J) \to \mathcal{B}(-1)^{\oplus s_\ell} \to 0.
\end{array}
$$

Corollary 6.2

Suppose that $n \geq 1$ and A is n-Goto with respect to Q. Then

- $\mathrm{e}_2(J) = n$ *if $d \geq 2$*
- $\mathrm{e}_i(J) = 0$ *for $3 \leq \forall\, i \leq d$, if $d \geq 3$.*

Exact sequences and Goto rings

When $\dim R = 1$, recall that $\mathfrak{c} = R : R[K]$.

- R is Gorenstein $\iff R = K$ ($\iff R = \mathfrak{c}$)
- R is AGL $\iff K/R \cong (R/\mathfrak{m})^{\oplus}$
- R is GGL $\iff K/R \cong (R/\mathfrak{c})^{\oplus}$
- R is 2-AGL $\implies K/R \cong (R/\mathfrak{c})^{\oplus} \oplus (R/\mathfrak{m})^{\oplus}$

When $\dim A = d > 0$,

- A is AGL $\iff \exists 0 \to A \to \mathrm{K}_A \to C \to 0$
  s.t. C is an Ulrich A-module with respect to $\mathfrak{m}$
- A is GGL $\implies \exists \sqrt{\mathfrak{a}} = m$ and $\exists 0 \to A \to \mathrm{K}_A \to C \to 0$
  s.t. C is an Ulrich A-module with respect to $\mathfrak{a}$

Note that

- $\mathfrak{m}^\ell \mathcal{S}_Q(J) = (0)$ for $\forall\, \ell \gg 0$
- $\mathfrak{m}^\ell \mathcal{S}_Q(J) = (0)$ for $\forall\, \ell \geq n$, if A is n-Goto with respect to Q
- $\mathfrak{m}^{n-1} \mathcal{S}_Q(J) \neq (0) \iff v(A/\mathfrak{c}) = 1$, if A is n-Goto and $\dim A = 1$.

Theorem 7.1

Suppose that $n \geq 1$, A is n-Goto with respect to Q, and $\mathfrak{m}^{n-1}\mathcal{S}_Q(J) \neq (0)$. Then $\exists \sqrt{\mathfrak{a}_i} = m$ with $\ell_A(A/\mathfrak{a}_i) = i$ and $\exists$ an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0 \ \text{ s.t. } \ C \cong \bigoplus_{i=1}^{n} M_i \ \text{ and } \ M_n \neq (0)$$

where M_i denotes an Ulrich A-module with respect to $\mathfrak{a}_i$ for $1 \leq \forall\, i \leq n$.

Thank you for your attention.